Efficient Computation of Hedge-Sensitivities via Automatic Differentiation

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Overview

- FastOpt
- Finite Difference solver for Black Scholes
- Greeks in forward mode of AD
- Greeks in reverse mode of AD
- Performance

FastOpt

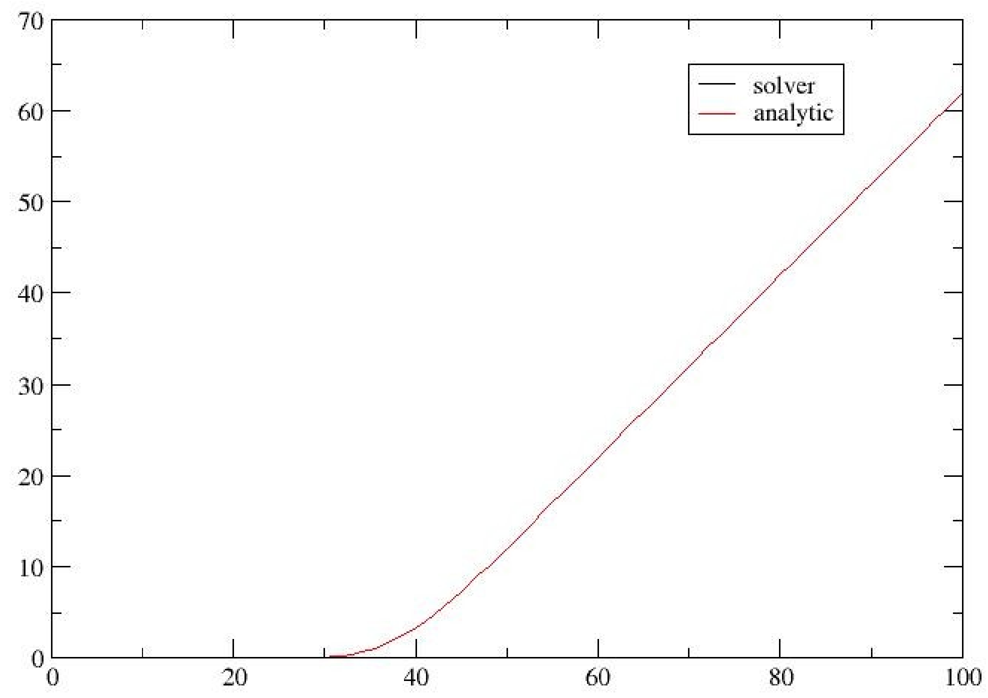
- SME founded in 2000 in Hamburg
- by R. Giering and T. Kaminski
- Michael Vossbeck joined in 2003
- Simon Blessing in 2007
- ~ 20 years of Experience in AD and data assimilation
- Two lines of business:
  - Development and marketing of automatic differentiation (AD) tools
  - Application project with/for our customers/project partners with focus on AD, inverse modelling, data assimilation...

Finite Difference Solver

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

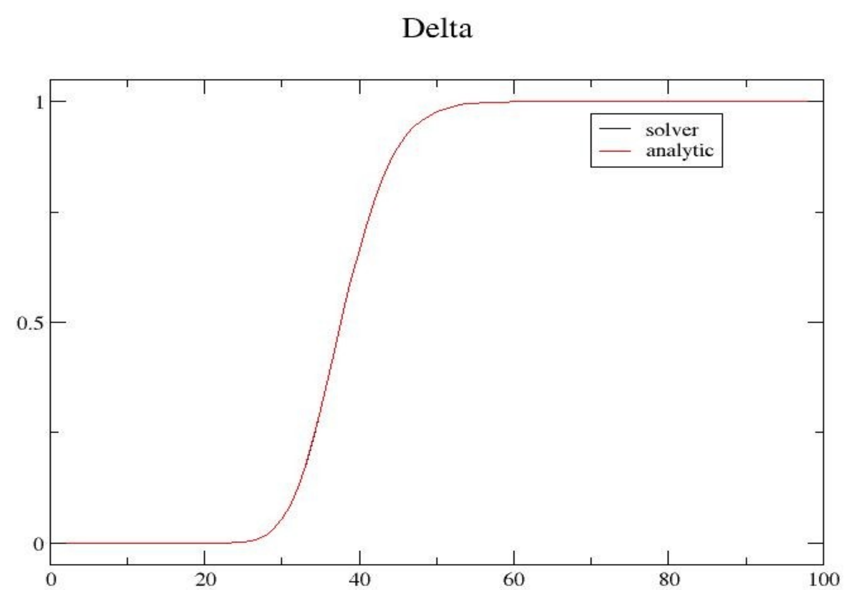
- Determines price of option (f) on one underlying (S) (Deutsch, 2001)
- Integrates Black-Scholes-Merton backwards in time (t) on a 101 x 401 grid in (S x t) space
- Example (Hull, 2002, chapter 11.8): European Call Option with
  - S=42
  - Strike K=40
  - Time to maturity T = 0.5 a
  - Volatility σ = 20% pa
  - Interest rate r = 10 % pa
- Payoff not differentiable: Use a sheared grid with centreline $S_c(t)$ moving from S at t=0 to K at t=T (details: tomorrow's talk by Alex Prideaux)

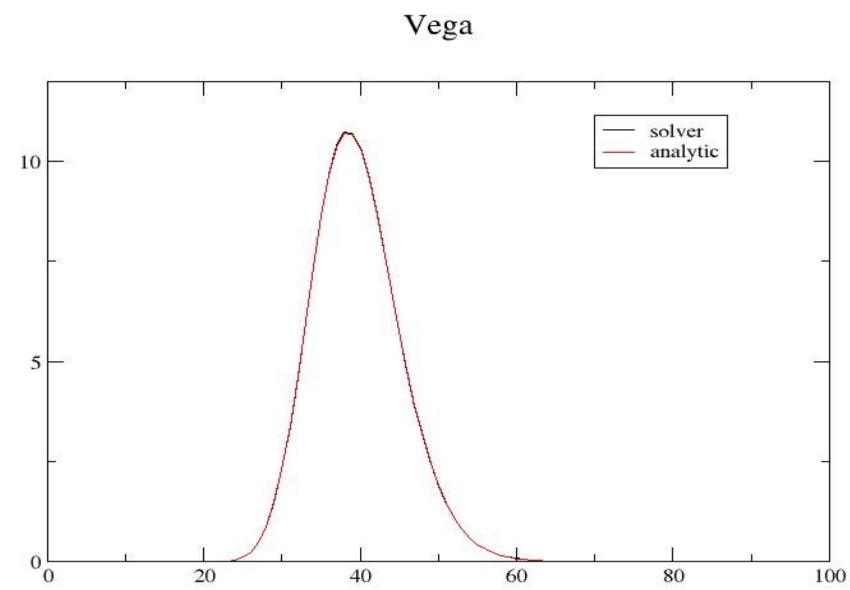
Value of European Call



Greeks for Hedging

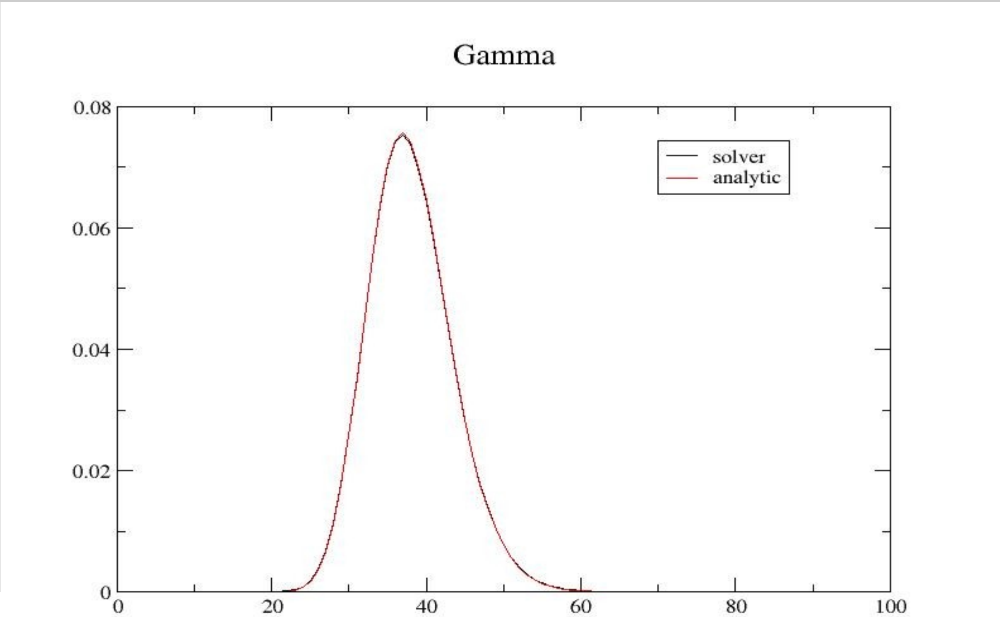
- Sensitivity of the option price with respect to changes in the individual influence factors, e.g.
  - Delta: change with respect ot change in S, the value of the underlying
  - Vega: change with respect to σ, the volatility
- Evaluate these sensitivities via AD of the solver code via TAF
- For only few control variables use forward mode, i.e. the (vector) tangent code of the solver

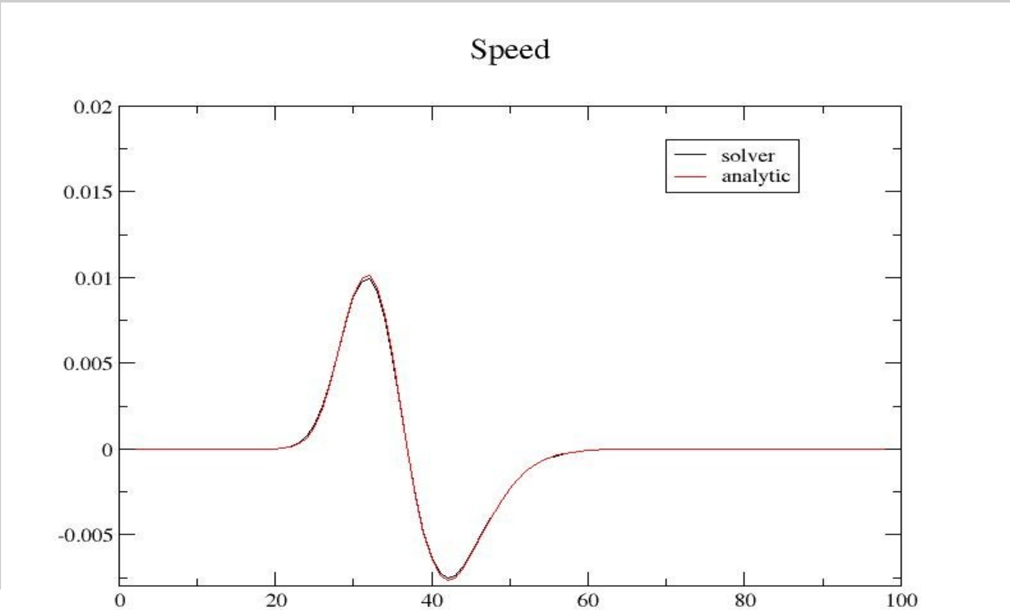




Greeks for Hedging

- What about higher order sensitivities?
  - Gamma: df^2/dS^2
  - Vanna: $df^2/dSd\sigma$
- Just need to apply TAF a second time to the tangent code it had generated for delta and vega
- Again the forward mode is most efficient
- That works also for three times, e.g. for Speed: df^3/dS^3



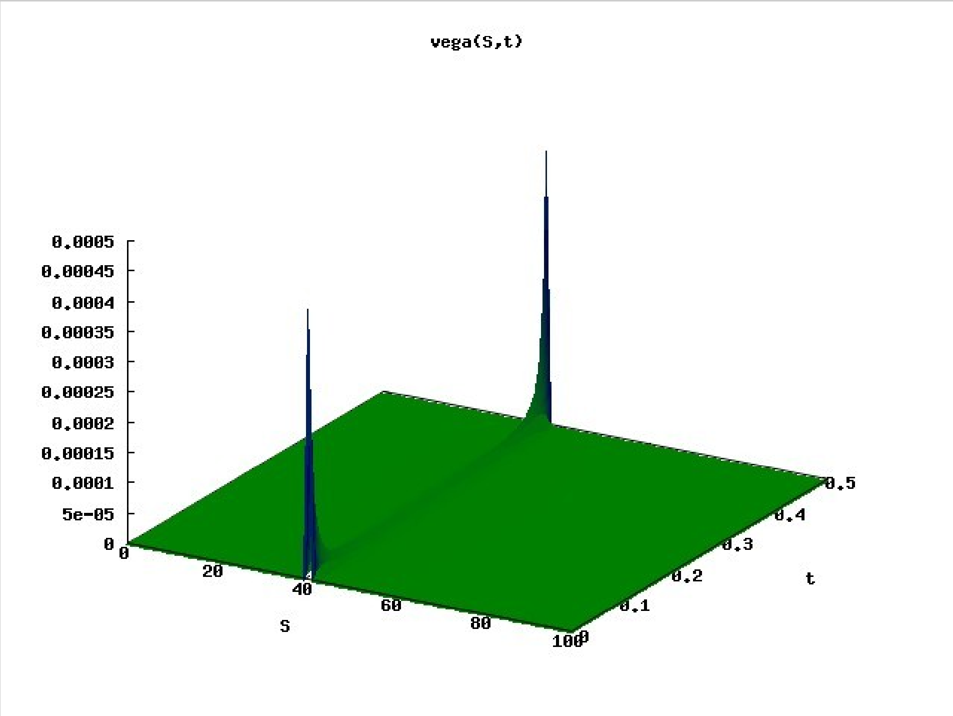


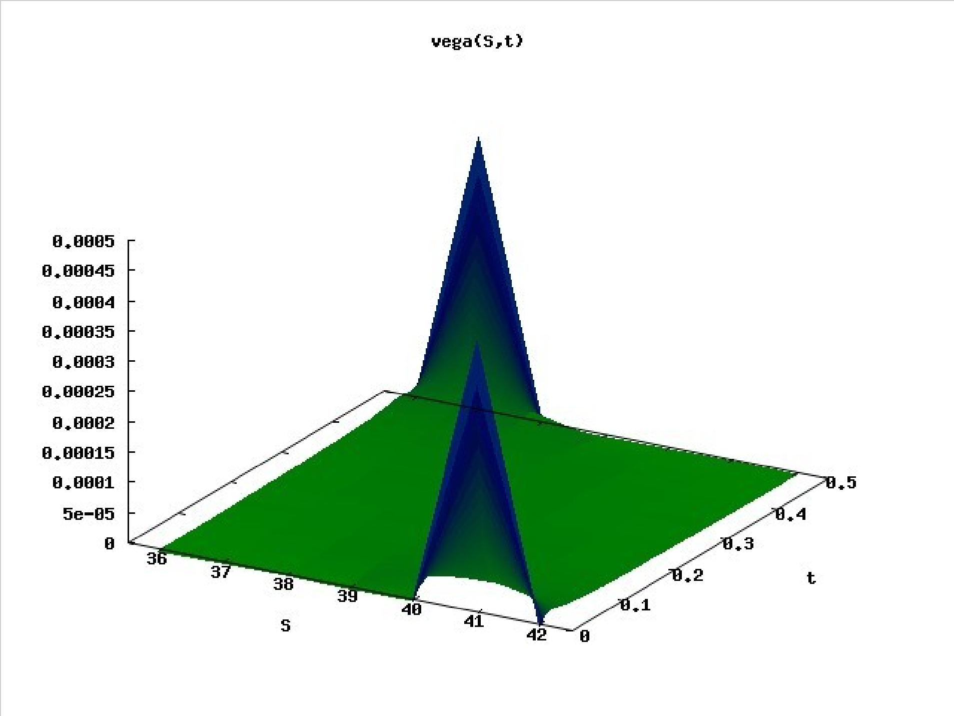
Greeks in reverse mode

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

- Actually σ is not constant (vola surface)
- σ(S,t) is defined by 101 x 401 values on the (S x t) grid
- Sensitivities of price to a change in each of these values: vega(S,t)?
- Also required for calibration of model to market prices
- Clearly a case for the reverse mode:
  - 101 x 401 independents
  - One dependent: option price
- No analytical solution
- Can check correctness of AD derivative, as usual

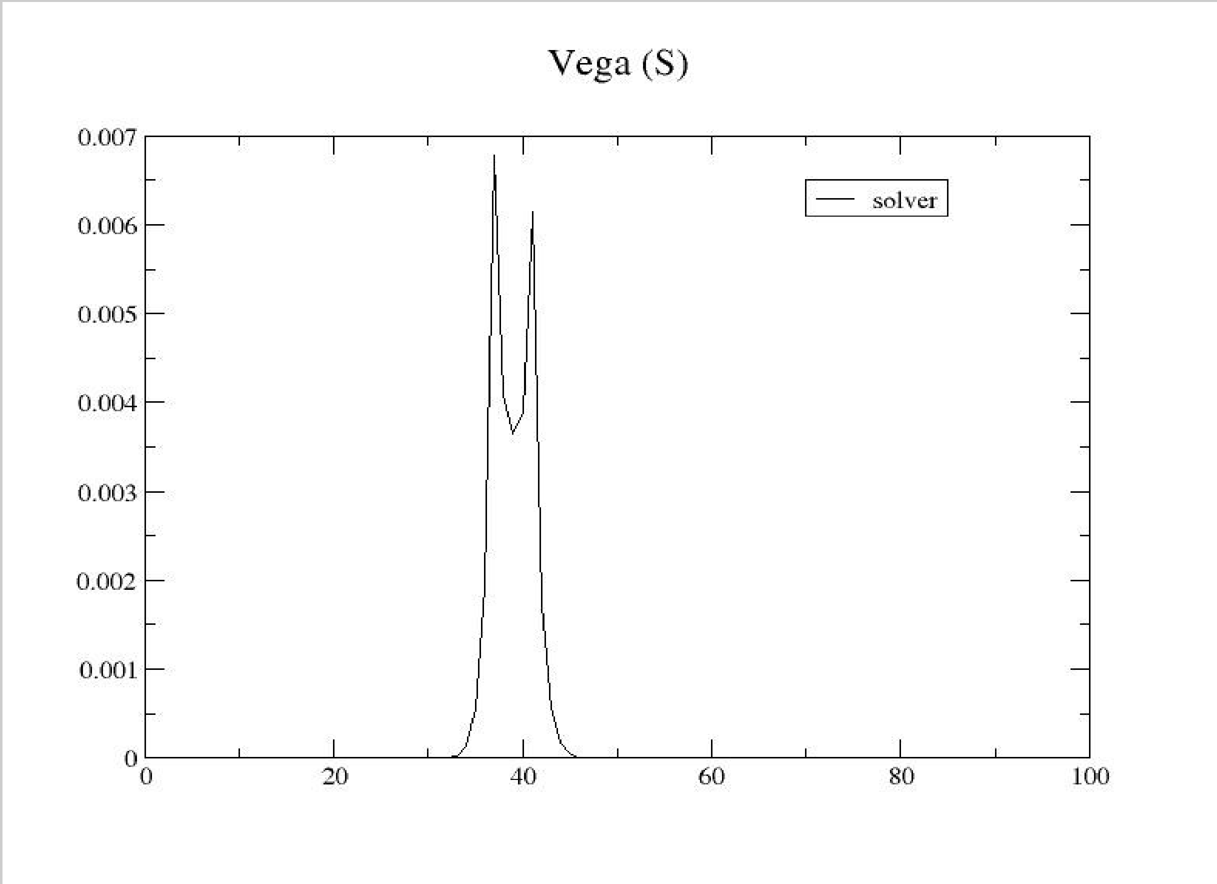
Greeks in reverse mode





*Fast*Opt

Greeks in reverse mode



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Verification

```
*****************************************************
CHECK OF ADM USING eps = 0.100E-05
*****************************************************
    i      j          x(i)          y(j)  delta f/eps        grad f      rel diff
   38     43 0.200000E+00 0.475372E+01 0.000000E+00 0.658669E-10 0.100000E+01DIFFERENT
  139     43 0.200000E+00 0.475372E+01 0.888178E-09 0.696721E-09 0.215562E+00DIFFERENT
  240     43 0.200000E+00 0.475372E+01 0.355271E-08 0.369038E-08 0.373041E-01DIFFERENT
  341     43 0.200000E+00 0.475372E+01 0.115463E-07 0.132677E-07 0.129740E+00DIFFERENT
  442     43 0.200000E+00 0.475372E+01 0.364153E-07 0.369520E-07 0.145226E-01DIFFERENT
  543     43 0.200000E+00 0.475372E+01 0.861533E-07 0.859609E-07 0.223312E-02DIFFERENT
  644     43 0.200000E+00 0.475372E+01 0.175859E-06 0.175182E-06 0.384931E-02DIFFERENT
  745     43 0.200000E+00 0.475372E+01 0.321521E-06 0.322870E-06 0.418084E-02DIFFERENT
  846     43 0.200000E+00 0.475372E+01 0.550671E-06 0.550165E-06 0.917600E-03
  947     43 0.200000E+00 0.475372E+01 0.879297E-06 0.880515E-06 0.138324E-02DIFFERENT
 1048     43 0.200000E+00 0.475372E+01 0.133848E-05 0.133905E-05 0.424333E-03
 1149     43 0.200000E+00 0.475372E+01 0.194955E-05 0.195198E-05 0.124517E-02DIFFERENT
 1250     43 0.200000E+00 0.475372E+01 0.274447E-05 0.274598E-05 0.548482E-03
 1351     43 0.200000E+00 0.475372E+01 0.374722E-05 0.374764E-05 0.111623E-03
 1452     43 0.200000E+00 0.475372E+01 0.498357E-05 0.498303E-05 0.109138E-03
 1553     43 0.200000E+00 0.475372E+01 0.647660E-05 0.647719E-05 0.914942E-04
 1654     43 0.200000E+00 0.475372E+01 0.825295E-05 0.825387E-05 0.110537E-03
 1755     43 0.200000E+00 0.475372E+01 0.103322E-04 0.103352E-04 0.288257E-03
 1856     43 0.200000E+00 0.475372E+01 0.127418E-04 0.127413E-04 0.384683E-04
 1957     43 0.200000E+00 0.475372E+01 0.154907E-04 0.154906E-04 0.969026E-05
 2058     43 0.200000E+00 0.475372E+01 0.185985E-04 0.185990E-04 0.294306E-04
 2159     43 0.200000E+00 0.475372E+01 0.220810E-04 0.220805E-04 0.228919E-04
 2260     43 0.200000E+00 0.475372E+01 0.259464E-04 0.259467E-04 0.131382E-04
 2361     43 0.200000E+00 0.475372E+01 0.302061E-04 0.302070E-04 0.316207E-04
 2462     43 0.200000E+00 0.475372E+01 0.348681E-04 0.348687E-04 0.163168E-04
 2563     43 0.200000E+00 0.475372E+01 0.399369E-04 0.399368E-04 0.399864E-05
 2664     43 0.200000E+00 0.475372E+01 0.454117E-04 0.454144E-04 0.597641E-04
 2765     43 0.200000E+00 0.475372E+01 0.513012E-04 0.513026E-04 0.277426E-04
 2866     43 0.200000E+00 0.475372E+01 0.576010E-04 0.576007E-04 0.532298E-05
 2967     43 0.200000E+00 0.475372E+01 0.643050E-04 0.643063E-04 0.204972E-04
 3068     43 0.200000E+00 0.475372E+01 0.714158E-04 0.714154E-04 0.521134E-05
 3169     43 0.200000E+00 0.475372E+01 0.789218E-04 0.789225E-04 0.903310E-05
 3270     43 0.200000E+00 0.475372E+01 0.868186E-04 0.868208E-04 0.257872E-04
 3371     43 0.200000E+00 0.475372E+01 0.951017E-04 0.951024E-04 0.717707E-05
 3472     43 0.200000E+00 0.475372E+01 0.103758E-03 0.103758E-03 0.330126E-05
 3573     43 0.200000E+00 0.475372E+01 0.112780E-03 0.112778E-03 0.144976E-04
 3674     43 0.200000E+00 0.475372E+01 0.122153E-03 0.122152E-03 0.757307E-05
 3775     43 0.200000E+00 0.475372E+01 0.131867E-03 0.131868E-03 0.583154E-05
 3876     43 0.200000E+00 0.475372E+01 0.141914E-03 0.141913E-03 0.438873E-05
 3977     43 0.200000E+00 0.475372E+01 0.152277E-03 0.152277E-03 0.509475E-05
```

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Computational Efficiency

- Run time of vector tangent with two independents is about run time of 1-3 solver integrations
- Run time of adjoint about 4 solver integrations
- Relative run times depend on compiler and options
- Solver code has about 200 lines w/o comments

Some derivative codes generated by TAF

Model	Model Reference	Lines	Language	Main Loop	TLM	ADM	comment	HES	AD References
NSC2KE	Mohamadi (1994)	2,500	F77	steady	2.4	3.4	iteration directive	9.8	Giering et al. (2005)
NS-Solver	Hinze (1999)	1,000	F77	steady	-	2.0	flow directives	-	Hinze and Slawig (2002)
IMBETHY	Knorr (2001)	5,000	F90	evolving	1.5	3.6	2 level checkpointing	12.7/5	Rayner et al. (2001)
MOM3	Pacanowski and Griffes (1999)	50,000	F77	evolving	yes	4.6	2 level checkpointing	-	Galanti et al. (2001)
MITGCM	Marshall et al. (1995)	100,000	F77	evolving	1.8	5.5	3 level checkpointing	11.0	Marotzke et al. (1999), Stammer (2000), Stammer et al. (2002), Heimbach et al. (2002), Stammer et al. (2002), Online Manual (2002)
Biomag code	Buecker et al. (2001)	83	F77		548/1098	3.1	-	-	Bischof et al. (2001)
NASA-DAO	Lin and Rood (1996), Lin and Rood (1997), Lin (1997)	87,000	F90	evolving	2.7	8.4	2 level checkpointing	-	Todling et al. (2003) Giering et al. (2005)
HB_AIRFOIL	Thomas et al. (2002), Thomas et al. (2002a), Hall et al. (2002a)	8,000	F90	steady+unsteady	yes	3	-	-	Thomas et al. (2003), Hall et al. (2002)
EULSOLDO	Cusdin and Müller (2003)	423	F77	-	2-3	2-3	AD restricted to kernel	-	Cusdin and Müller (2003), Cusdin and Müller (2003), Cusdin (2003)
3D N-S CFD code	Moinier et al. (2002)	699	F77	steady	1.2-3.4	1.2-3.4	flow and iteration directives	-	Cusdin and Müller (2003)
Rot-Disk	Beeson (2001), Timoshenko and Goodier (1970)	289	F77	-	7/6	7/6	iteration directive, (yet faster with structured AD)	-	Akkaram et al. (2003)
ARPS	Xue et al. (2000 , 2001)	40000	F90	evolving	2	11	2-level checkpointing	-	Xiao et al. (2004)
NAST2D	Griebel et al. (1998)	2700	F90	steady	90/289	1.8	iteration directive	-	-
NAST2D in 3D	Griebel et al. (1998)	3500	F90	steady	1.4	1.8	iteration directive	-	Othmer et al. (2006) and Kaminski et al. (2006)
NIRE-CTM	Taguchi (1993 , 1996)	860	F77	evolving	1.0	1.5	-	-	Taguchi (2005)

- Lines: total number of C code lines *without* comments
- Numbers for TLM and ADM give CPU time for (function + gradient) relative to forward model
- HES format: CPU time for Hessian * n vectors relative to forward model / n

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Derivative Codes generated by TAC++

Model	Model Reference	Lines	Language	Main Loop	TLM	ADM	comment	HES	AD References
LIBOR	Brace et al. (1997)	approx. 100	C	-	5.4/80	3.7	-	-	Giles (2007)
Two-stream	Pinty et al. (2006)	approx. 330	C	-	1.7	3.8	-	23/7	Lavergne et al. (2006), Pinty et al. (2007)
ROF (Computer Vision)	Pock et al. (2007)	approx. 60	C	-	1.6	1.9	-	-	Pock et al. (2007)
TAUij (1 core routine)	Gerhold (2005)	130	C	-	-	2.3	-	-	Voßbeck et al. (2008)
EULSOLDO	Cusdin and Müller (2003)	140	C	-	2.2	3.2	-	-	Voßbeck et al. (2004 , 2008)

Table is available on http://www.fastopt.com/products/tac/applications.html

- Lines: total number of C code lines *without* comments
- Numbers for TLM and ADM give CPU time for (function + gradient) relative to forward model
- HES format: CPU time for Hessian * n vectors relative to forward model / n

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Summary/Conclusions

- Used TAF to generate derivative code of a finite difference solver
- Provided Greeks up to third order, accurate sensitivites of the solver
- Useful for both hedging of a portfolio and calibration of the model
- Computation is “smoking” fast
- Fully automated derivative generation simplifies the maintenance of a modelling system for hedging/calibration